Markov process :

If $\{X(t), t \in T\}$ is a S.P such that given value $X(S)$, the values of $X(t)$, $t > S$ donot depend on the value of $X(u)$, $u < S$ then stochastic process is called as Markov process.

If for $t_1 < t_2 < \dots < t_n < t_{n+1}$

$$P\{a \le X(t_{n+1}) \le b / X(t_1) = x_1 ; X(t_2) = x_2, \dots X(t_n) = x_n\}$$

$$= P\{a \le X(t_{n+1}) \le b / X(t_n) = x_n\}$$

then the S.P $\{X(t), t \in T\}$ is a Markov process.

Note :

If the state space x is discrete then the Markov process is known as "Markov chain"

(or)

$$P[X_t = a]$$

$$P[X_0 = a_0]$$

$$P[X_1 = a_1 / X_0 = a_0]$$

$$P[X_n = a_n / X_0 = a_0 \dots X_{n-1} = a_{n-1}]$$

then $X_n = a_n$ is purely depends on the current value $X_{n-1} = a_{n-1}$, then

$$P\left[x_n = a_n / x_{n-1} = a_{n-1}\right]$$

when it is true the S.P is called Markov process. In Markov process if the state space is discrete then it is known as Markov chain.

Markov chain :

observations are recorded in a chronological sequence. eg., population of a country.

Independence and dependence of an observations are the concepts used in the study of observation. The concepts independence is a +ive one and the dependence is vague

The dependence x_{n+1} doesnot extend beyond its predecession x_n.

The S.P $\{x_n ; n \geq 1\}$ is a Markov chain (or) Markov dependent if for all $i_0, i_1, \ldots i_{n+1} \in I$ and $\forall\, n$

$$P\left[x_{n+1} = i_{n+1} / x_0 = i_0, x_1 = i_1, \ldots x_n = i_n\right] =$$

$$P\left[x_{n+1} = i_{n+1} / x_n = i_n\right]$$

ie Markov property implies that future value depends only under the present value not on the past values.

Transition probabilities :

The probability of moving from the state i to the state j in one unit time period $(t=1)$ is denoted by $P_{ij}^{(1)}$

It is the basic concept used to study the structure of the Markov chain. If the transition probabilities independent of n, then the Markov chain is said to be homogeneous with respect to time

If the transition probabilities depends on n, then the Markov chain is called as non-homogeneous. By one-step transition probl. $P_{ij}^{(1)}$, we mean that the probl. of moving from the state i at the n^{th} period to the state j at the $(n+1)^{th}$ period

(ie) $P_{ij}^{(1)} = P\left[X_{n+1} = j / X_n = i\right]$

It is concerned with the consequtive points $n, n+1$.

If it is concerned with two non-consecutive points n, $n+m$ then the transition probl.

$$P_{ij}^{(m)} = P\left[X_{n+m} = j / X_n = i\right]$$

is called as m-step transition probl.

when $m = 1$, $P_{ij}^{(1)}$ is called as one-step transition probl.

Stochastic Matrix (or) P - Matrix :

An arrangement of transition probabilities in a matrix form is called as stochastic matrix when the transition probl. P_{ij} satisfy the following condition

(i) $P_{ij} \geq 0$.

(ii) $\sum_j P_{ij} = 1$; $\forall\ i = 1, 2, \ldots n$

$$P = \begin{array}{c c} & \begin{array}{cccccc} A_1 & A_2 & \cdots & A_j & \cdots & A_n \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_i \\ \vdots \\ A_n \end{array} & \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1j} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2j} & \cdots & P_{2n} \\ \vdots & & & & & \\ P_{i1} & P_{i2} & \cdots & P_{ij} & \cdots & P_{in} \\ \vdots & & & & & \\ P_{n1} & P_{n2} & \cdots & P_{nj} & \cdots & P_{nn} \end{bmatrix} \end{array}$$

when P_{ij} is the transition probl. that a customer preference will switch from brand 'i' to brand 'j' from one period to the next period and sum of P_{ij} equal to one. ie $\sum P_{ij} = 1 \; \forall \; i = 1, 2, \ldots n$

Example:

Formulate the transition probl. matrix of a Markov chain based on the following data.

Census data divide the house holds into two groups namely economically stable and economically depressed population.

Over a 10 years period the probl. of a stable household remaining stable is 0.92, while the prob. of stable household become economically depressed is 0.08. The prob. of economically depressed household becomes stable is 0.03, while the prob. of economically depressed household remaining depressed is 0.97.

Solution:

Let the state of the Markov chain be represented by 1 for economically stable and by 0 for economically depressed.

The transition probabilities are given by

$$P = \begin{array}{c c} & \begin{array}{cc} & j \\ ES & ED \\ 1 & 0 \end{array} \\ i \begin{array}{cc} ES & 1 \\ ED & 0 \end{array} & \begin{bmatrix} 0.92 & 0.08 \\ 0.03 & 0.97 \end{bmatrix} \end{array}$$

$P_{11} = P\{X_1 = 1 / X_0 = 1\} = 0.92$

$P_{10} = P\{X_1 = 0 / X_0 = 1\} = 0.08$

$P_{01} = P\{X_1 = 1 / X_0 = 0\} = 0.03$

$P_{00} = P\{X_1 = 0 / X_0 = 0\} = 0.97$.

Then the transition prob. matrix of the two states markov chain is formed as

$$P = \begin{bmatrix} 0.92 & 0.08 \\ 0.03 & 0.97 \end{bmatrix}$$

Example:

A single coin tossing experiment is repeated for no. of times. The possible outcome in trial are head and tail with p & q respectively $\Rightarrow p + q = 1$

A r.v X_n is defined as

$$X_n = \begin{cases} 1 & ; \text{if } n^{th} \text{ trial results in head} \\ 0 & ; \text{if } n^{th} \text{ trial results in tail} \end{cases}$$

Then the collection $\{X_n, n \geq 1\}$ is a S.P with state space $\{0, 1\}$

Let $S_n = X_1 + X_2 + \cdots + X_n$ be the r.v representing the no. of heads in the first 'n' trial. The possible values taken by S_n is

$$S_n = \{0, 1, \ldots, n\}$$

and it constitutes the state space of the process.

The S.P $\{S_n, n \geq 1\}$ is a Markov chain because

$$P\{S_{n+1} = j / S_n = i\} = P_{ij} \quad \forall\, i, j = 0, 1, 2, \ldots$$

where

$$P_{ij} = \begin{cases} p & ; \text{if } j = i+1 \\ q & ; \text{if } j = i \\ 0 & ; \text{otherwise} \end{cases}$$

Then the transition probability matrix of the Markov chain is formed as

$$
\begin{array}{cc}
 & \text{$S_{n+1} = j$} \\
 & \begin{array}{cccccccc} 0 & 1 & 2 & \cdots & r & r+1 & \cdots \end{array} \\
\begin{array}{c} 0 \\ S_n = i \quad 1 \\ \vdots \\ p_{ij} = \quad \vdots \\ r \\ \vdots \end{array} &
\begin{bmatrix}
q & p & 0 & \cdots & 0 & 0 & \cdots \\
0 & q & p & \cdots & 0 & 0 & \cdots \\
\vdots & & & & & & \\
\vdots & & & & & & \\
0 & 0 & 0 & \cdots & q & p & \cdots \\
\vdots & \vdots & & & & \vdots &
\end{bmatrix}
\end{array}
$$

where $p+q=1$

Higher order transition probabilities:

In general, a higher order T.P can be defined as

$$P\left[X_{m+n}=j \,/\, X_m = i\right] = P_{ij}^{(n)}$$

which means that the process have been started from state i at m^{th} step reduce the state j in n^{th} steps.

Higher order transition probability matrix:

The n^{th} step t.p.m can be obtained as $P^n = P^{n-1} \cdot P$

Proof:

consider the two step t.P

$$P_{ij}^{(2)} = \sum_{k \in S} P_{ik} P_{jk} \quad \text{(by chapman kolmogorov equation)}$$

$$= \begin{bmatrix} p_{i0} & p_{i1} & \cdots & p_{in} \end{bmatrix} \begin{bmatrix} \cdots & p_{0j} & \cdots \\ \cdots & p_{1j} & \cdots \\ & \vdots & \\ \cdots & p_{nj} & \cdots \end{bmatrix}$$

$$\Rightarrow P^2 = \left[p_{ij}^{(2)} \right] = P \cdot P$$

ie 2^{nd} order t.p.m is the product of two one-step t.p.

Extending the above ~~xxx~~ to n-step t.p

$$p_{ij}^{(n)} = \sum_{k \in S} p_{ik}^{(n-1)} \, p_{kj}$$

$$P^n = p_{ij}^{n}$$

$$P^n = P^{n-1} \cdot P$$

Derivation of chapman kolmogorov Equation

Let us consider the one-step t.p of a M.C

ie $p_{ij}^{(1)} = P\left[X_n = j / X_{n-1} = i \right]$

$\Rightarrow$ The prob. of the outcomes at the n^{th} step given the outcome of the previous $(n-1)^{th}$ step.

Similarly, the m-step t.p of the M.C is

$$P_{ij}^{(m)} = P\left[X_{n+m} = j / X_n = i\right]$$

$\Rightarrow$ The prob. of the outcome at $(n+m)^{th}$ step given the outcome at the n^{th} step.

(ie) The prob. of the transition from the state i to the state j in exactly m-steps.

Let us consider the two-step t.p of the M.C as

$$P_{ij}^{(2)} = P\left[X_n = j / X_{n-2} = i\right]$$

(ie) The prob. of the transition from the state i to the state j takes exactly two steps.

(ie) $$P_{ij}^{(2)} = P\left\{X_n = j ; X_{n-1} = r / X_{n-2} = i\right\} \quad \text{--- (1)}$$

$$= P\left\{X_n = j / X_{n-1} = r, X_{n-2} = i\right\} . P\left\{X_{n-1} = r / X_{n-2} = i\right\}$$

$$= P\left\{X_n = j / X_{n-1} = r\right\} . P\left\{X_{n-1} = r / X_{n-2} = i\right\}$$

$$P_{ij}^{(2)} = P_{rj}^{(1)} . P_{ir}^{(1)}$$

$= P_{ir}^{(1)} P_{rj}^{(1)}$ is one step t.p for fixed value of r. Since r takes values $r = 1, 2, \ldots$ inbetween i & j ; $i < j$

$$P_{ij}^{(2)} = P\left\{X_n = j,\ X_{n-1} = r \,/\, X_{n-2} = i\right\}$$

$$= P\left[\left\{X_n = j,\ X_{n-1} = 1\right\} \cup \left\{X_n = j,\ X_{n-1} = 2\right\} \cup \ldots \right.$$

$$\left. \ldots \,/\, X_{n-2} = i\right\}\Big]$$

$$= \sum_r P\left\{X_n = j,\ X_{n-1} = r \,/\, X_{n-2} = i\right\} \quad \text{using } (1)$$

$$P_{ij}^{(2)} = \sum_r P_{ir}^{(1)} P_{rj}^{(1)}.$$

IIIly,

$$P_{ij}^{(m+1)} = P\left\{X_{n+m+1} = j \,/\, X_n = i\right\}$$

$$= P\left\{X_{n+m+1} = j,\ X_{n+m} = r \,/\, X_n = i\right\}$$

$$= P\left\{X_{n+m+1} = j \,/\, X_{n+m} = r,\ X_n = i\right\} \cdot P\left\{X_{n+m} = r \,/\, X_n = i\right\}$$

$$= P\left\{X_{n+m+1} = j \,/\, X_{n+m} = r\right\} \cdot P\left\{X_{n+m} = r \,/\, X_n = i\right\}$$

$$= P_{rj}^{(1)} P_{ir}^{(m)}$$

$$= P_{ir}^{(m)} P_{rj}^{(1)}, \quad \text{for fixed value of } r$$

If r takes the different values (mutually exclusive) such as $r = 1, 2, \ldots$ then

$$P_{ij}^{(m+1)} = \sum_r P_{ir}^{(m)} P_{rj}^{(1)}$$

In general, we get for a M.C with stationary t p as

$$P_{ij}^{(m+k)} = P\{X_{n+m+k} = j / X_n = i\}$$

$$= P\{X_{n+m+k} = j, X_{n+m} = r / X_n = i\}$$

$$= P\{X_{n+m+k} = j / X_{n+m} = r, X_n = i\} . P\{X_{n+m} = r / X_n = i\}$$

$$= P\{X_{n+m+k} = j / X_{n+m} = r\} . P\{X_{n+m} = r / X_n = i\}$$

$$= P_{rj}^{(k)} P_{ir}^{(m)}$$

$$= P_{ir}^{(m)} P_{rj}^{(k)}, \text{ for fixed value of } r.$$

when r takes different values such as $r = 1, 2, \ldots$ then

$$P_{ij}^{(m+k)} = \sum_r P_{ir}^{(m)} P_{rj}^{(k)}$$

which is known as chapman - kolmogorov equation.

Let $P = \{P_{ij}\}$ be the unit - step t.p.m of a M.C

$P^{(m)} = \{P_{ij}^{(m)}\}$ in m - step t.p.m of a M.C

$m = 2$, $P^{(2)} = \{P_{ij}^{(2)}\}$

$$= P . P = P^2$$

$$\therefore P^{m+1} = P^{(m)} . P$$

$$\text{|||ly} \quad P^{m+n} = P^{(m)} . P^{(n)}$$

Thus the problem of obtaining m-step t.P is reduced to the problem of calculating the m^{th} power of P.

Note:

when the M.C is finite then the chapman's kolmogorov equ. is used to compute the higher step t.p.m.

Problem:

Find the 3^{rd} step t.p of the M.C when its unit step t.P is given as

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix} \end{array}$$

Solution:

Given that the unit step t.p.m of a M.C

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$P^{(2)} = P.P = \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$P^{(2)} = \begin{pmatrix} 1/4 + 1/6 & 1/4 + 2/6 \\ 1/6 + 2/9 & 1/6 + 4/9 \end{pmatrix}$$

$$P^{(2)} = \begin{pmatrix} 10/24 & 14/24 \\ 21/54 & 33/54 \end{pmatrix}$$

$$P^{(3)} = P^{(2+1)} = P^{(2)} \cdot P$$

$$= \begin{pmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$P^{(3)} = \begin{pmatrix} 5/24 + 7/36 & 5/24 + 14/36 \\ 7/36 + 11/54 & 7/36 + 22/54 \end{pmatrix}$$

$$= \begin{pmatrix} 29/72 & 43/72 \\ 43/108 & 65/108 \end{pmatrix}$$

By using chapman kolmogorov equ. for two-step transition probabilies

$$P^{(2)} = \sum_{r=0}^{1} P_{ir}^{(1)} P_{rj}^{(1)}$$

$$= P_{i0}^{(1)} P_{0j}^{(1)} + P_{i1}^{(1)} P_{1j}^{(1)}$$

If $i = j = 0$,

$$P_{00}^{(2)} = P_{00}^{(1)} P_{00}^{(1)} + P_{01}^{(1)} P_{10}^{(1)}$$

$$= (1/2)(1/2) + (1/2)(1/3) = 5/12$$

$i = 0,\ j = 1,$

$P_{01}^{(2)} = 1 - P_{00}^{(1)} = 1 - 5/12 = 7/12.$

$i = 1,\ j = 0,$

$P_{10}^{(2)} = P_{10}^{(1)} P_{00}^{(1)} + P_{11}^{(1)} P_{10}^{(1)}$

$= (1/3)(1/2) + (2/3)(1/3)$

$= 7/18$

$i = j = 1,$

$P_{11}^{(2)} = 1 - P_{10}^{(2)} = 1 - 7/18 = 11/18$

$$\therefore\ P^{(2)} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{pmatrix} \end{array}$$

$P^{(3)} = P^{(2+1)} = \sum_{r=0}^{1} P_{ir}^{(2)} P_{rj}^{(1)}$

$= P_{i0}^{(2)} P_{0j}^{(1)} + P_{i1}^{(2)} P_{1j}^{(1)} \quad ; r = 0, 1$

$i = j = 0,$

$P_{00}^{(3)} = P_{00}^{(2)} P_{00}^{(1)} + P_{01}^{(2)} P_{10}^{(1)}$

$= (5/12)(1/12) + (7/12)(1/3)$

$= 29/72$

$i = 0, \; j = 1,$

$$P^{(3)}_{01} = 1 - P^{(3)}_{00} = 1 - 29/72 = 43/72.$$

$i = 1, \; j = 0,$

$$P^{(3)}_{10} = P^{(2)}_{10} P^{(1)}_{00} + P^{(2)}_{11} P^{(1)}_{10}$$

$$= \left(7/18\right)\left(1/2\right) + \left(11/18\right)\left(1/3\right)$$

$$= 43/108$$

$i = j = 1,$

$$P^{(3)}_{11} = 1 - P^{(3)}_{10} = 1 - 43/108 = 65/108$$

$$\therefore P^{(3)} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 29/72 & 43/72 \\ 43/108 & 65/108 \end{pmatrix} \end{array}$$

which is the 3^{rd} step t.p of the M.C.

Classification of States:

(i) Accessible:

The state 'j' is said to be accessible from the state 'i' iff there exists a +ve no. such that $P^{(n)}_{ij} > 0$

(ie) State j is accessible from state i if there is +ve prob. that in a finite no. of transitions state j can be reached starting from state i, It can be represented by

$$i \to j \, , \; P_{ij}^{(n)} > 0 \; ; \; n \geq 1$$

$$j \to i \, , \; P_{ji}^{(n)} > 0 \; ; \; n \geq 1$$

Every state is accessible from itself since $P_{ii}^{(0)} = 1$.

(ii) communicate:

Two states i & j, each accessible to other are said to be communicate and we can write it as $i \leftrightarrow j$.

If two state i & j donot communicate then either

$$P_{ij}^{(n)} = 0 \;\; \forall \; n \geq 0$$

(or)

$$P_{ji}^{(n)} = 0 \;\; \forall \; n \geq 0$$

(or)

both relations are true

(a) Reflexivity:

$i \leftrightarrow i$, where ~~the~~ ~~same~~ state starts, it ends there, a consequence of the definition of

$$P_{ij}^{(0)} = \delta_{ij} = \begin{cases} 1 & ; i=j \\ 0 & ; i \neq j \end{cases}$$

(b) Symmetry:

If $i \leftrightarrow j$, then $j \leftrightarrow i$, from the definition of communication.

(c) Transitivity:

If $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$.

If the above three states will satisfy with each other then it is equivalence class.

(iii) Absorbing State:

$P_{ii} = 1$, i is said to be absorbing state

$$f_{ii}^{(n)} = P\left[X_n = i, X_v \neq i \text{ for } v = 1, 2, \ldots n-1 / X_0 = i\right]$$

having ~~started~~ started from i^{th} state, the process reaches the state i at the n^{th} step for the 1^{st} time is called First Return Probability

$$f_{ij}^{(n)} = P\left[X_n = j, X_v \neq j \text{ for } v = 1, 2, \ldots (n-1) / X_0 = i\right]$$

having started from i^{th} state, the process reaches the state j at the n^{th} step ~~from~~ for the 1^{st} time is called First Passage Probability

(iv) Recurrent (or) persistent state:

We say a state i is recurrent if and only if $\sum_{n=1}^{\infty} f_{ii}^{n} = 1$. This says that a state i is recurrent iff starting from state i the prob. of returning to state i after some finite length of time is one.

A non-recurrent state is also said to be transient state.

(ie) $\sum f_{ii}^{(n)} = 1 \longrightarrow$ recurrent

$\sum n f_{ii}^{(n)} = \mu_{ii} \longrightarrow$ Mean recurrent time

If $\mu_{ii} < \infty$, then the state is +ve (non-null) recurrent.

If $\mu_{ii} = \infty$, then the state is null recurrent

(v) Transient state:

The state j is said to be transient if $f_{jj} < 1$

(ie) The prob. of the first (or) ultimate return to the state j has value less than one. If mean recurrence time $\mu_{jj} = \infty$, then the state is also called as transient state.

(vi) Periodic State:

The state j is called periodic if a return to the state j is possible only in steps of integral multiplier of a certain integer 't'

$$P_{jj}^{(n)} = 0 \quad \text{for} \quad n = kt, \; t = 1, 2, \ldots$$

The largest integer 't' which has this property is called as the periodic state.

If the state j has the recurrent and j cannot be reached except at $t, 2t, 3t, \ldots$ steps. Then the state j is called periodic state with period 't', $\forall\, t \geq 2$

A state j is called as aperiodic if it has a period one $(t = 1)$.

(vii) Ergodic State:

The state j is said to be Ergodic state if it is recurrent, non-null and aperiodic state.

Example: Let $\{X_n, n \geq 1\}$ be a M.C with t.p.m as

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 2/3 & 1/3 & 0 & 0 & 0 \\ 2 & 1/3 & 0 & 1/3 & 1/3 & 0 & 0 \\ 3 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 4 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 5 & 0 & 0 & 0 & 1/3 & 2/3 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Classify the states of the M.C.

Sol^n

The t.p.m of a M.C is given as

$$
\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} &
\begin{bmatrix}
0 & 2/3 & 1/3 & 0 & 0 & 0 \\
1/3 & 0 & 1/3 & 1/3 & 0 & 0 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
0 & 0 & 0 & 1/2 & 1/2 & 0 \\
0 & 0 & 0 & 1/3 & 2/3 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

$\left.\begin{array}{l} P_{12} = 2/3 > 0 \Rightarrow 1 \to 2 \\ P_{13} = 1/3 > 0 \Rightarrow 1 \to 3 \end{array}\right\} \Rightarrow$ State 2 & 3 are accessible from the State 1.

$\left.\begin{array}{l} P_{12} = 2/3 > 0 \\ P_{21} = 1/3 > 0 \end{array}\right\} 1 \longleftrightarrow 2 \Rightarrow$ The states 1 & 2 are said to be communicative states

$\left.\begin{array}{l} P_{13} = 1/3 > 0 \\ P_{31} = 1/6 > 0 \end{array}\right\} 1 \longleftrightarrow 3 \Rightarrow$ States 1 & 3 are communicative

$\left.\begin{array}{l} P_{45} = 1/2 > 0 \\ P_{54} = 1/3 > 0 \end{array}\right\} 4 \longleftrightarrow 5 \Rightarrow$ State 4 & 5 are communicative

$P_{66} = 1 \Rightarrow$ State 6 is said to be absorbing

Example:

Classify the states
$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1/2 & 1/2 \\ 1 & 1/2 & 0 & 1/2 \\ 2 & 1/2 & 1/2 & 0 \end{array}$$

soln

$$\left.\begin{array}{l} P_{01} = 1/2 > 0 \\ P_{10} = 1/2 > 0 \end{array}\right\} \Rightarrow 0 \longleftrightarrow 1 \text{ --- } (1)$$

$$\left.\begin{array}{l} P_{12} = 1/2 > 0 \\ P_{21} = 1/2 > 0 \end{array}\right\} \Rightarrow 1 \longleftrightarrow 2 \text{ --- } (2)$$

from (1) & (2)

$$0 \longleftrightarrow 2 \text{ --- } (3)$$

from (1), (2) & (3) all the states are communicating with each other

$\Rightarrow$ M.C is irreducible.

Irreducible Markov chain:

If every state in a M.C can be reach from every other state from any no. of transition then the M.C is said to "irreducible".

(ie) A process is irreducible if all states communicate with each other.

Example:

The t.p.m of the M.C with states 0, 1, 2 is

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

The given M.C is a irreducible because every state can be reached from any other state at any no. of transition.

Soln:

$$\text{Given that } P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$P^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

$$P^3 = P^2 . P^1 = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} = P$$

$$P^4 = P^3 . P^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} = P^2$$

In general,

$$P^{(2n)} = P^{(2)}$$

$$P^{(2n+1)} = P^{(1)}$$

$$\Rightarrow P_{ii}^{(2n)} > 0 \quad \text{for each } i$$

$$P_{ii}^{(2n+1)} = 0 \quad \text{for each } i$$

The states are periodic with period 2

ie $t = 2$.

Persistent State

The state j is said to be persistent state if $f_{jj} = 1$

$$f_{jj} = \sum_n f_{jj}^{(n)}$$

Let $j = 1$, $f_{11} = 1 \Rightarrow$ State 1 is called as persistent.

$$f_{11} = \sum f_{11}^{(n)} = f_{11}^{(1)} + f_{11}^{(2)}$$

$$f_{11}^{(1)} = P_{11}^{(1)} = 0$$

$$f_{11}^{(2)} = P_{11}^{(2)} = 1$$

$$\therefore f_{11} = f_{11}^{(1)} + f_{11}^{(2)} = 0 + 1 = 1.$$

$\therefore f_{11} = 1 \Rightarrow i$ is a persistent state

Let $j = 0$

$$f_{00} = \sum f_{00}^{(n)}$$

$$= f_{00}^{(1)} + f_{00}^{(2)}$$

$$= 0 + 1/2 = 1/2$$

$$f_{00} = 1/2 < 1$$

$\therefore f_{00} < 1 \Rightarrow$ State 0 is called transient state.

Let $j = 2$

$$f_{22} = \sum f_{22}^{(n)}$$

$$= f_{22}^{(1)} + f_{22}^{(2)}$$

$$= 0 + 1/2 = 1/2 < 1$$

$\therefore f_{22} = 1/2 < 1 \Rightarrow$ state 2 is called transient state.

Hence 0 & 2 are transient state and 1 is persistent.

For the state i,

$$\mu_{ii} = \sum n \cdot f_{ii}^{(n)} \Rightarrow \text{Mean recurrent time}$$

Let $i = 1$,

$$\mu_{11} = \sum n f_{11}^{(n)}$$

$$= (1) \cdot f_{11}^{(1)} + (2) \cdot f_{11}^{(2)}$$

$\mu_{11} = 0 + 2(1) = 2 < \infty$ (non-null)

$$P_{11}^{(2n)} = \frac{t}{\mu} = \frac{2}{2} = 1 \quad \text{as } n \to \infty.$$

State 1 is called as persistent non-null with period 2.

Limiting Behaviour of a n-step transition probability :

Some results on finite M.C

Statement (i) :

For a finite irreducible and aperiodic M.C for all $n \geq N$, then n-step prob Matrix consists no zero elements.

Proof :

The chain is finite and irreducible. Therefore all the states communicate with each other. Then by def. for any states $i, j \in S$ there exists ($\exists$) an integer $n(i,j)$ such that

$P_{ij}^{n(i,j)} > 0$ [Prob. that the states reaches in n-step is greater than zero]

The chain is aperiodic and above is true for all $n(i,j) \geq N(i,j)$ [after no. of steps]

Now if we consider

$$M = \max_{i,j} \{ n(i,j) + N(i,j) \}$$

Then for any $n \geq M$, $P_{ij}^{(n)} > 0$.

Statement (ii):

For a finite M.C as the no. of steps tends to infinite, the prob that the process is in a transient state is always zero. Irrespective of the state from where it starts.

ie $\lim_{n\to\infty} P_{ij}^{(n)} \to 0$

where j being a transient state and i being either transient or recurrent

Proof:

Consider $\lim_{n\to\infty} P_{ij}^{(n)}$, $j \in T$ [T denotes set of transient state] where i is recurrent, since return to a recurrent state is definite the above result follows obviously.

Suppose i also belongs to the set of transient states ($i \in T$) then $\exists$ some +ve prob. for the process to come out of transient set (T) in some steps. Let 'n' be the maximum no. of least steps required to happen this.

Since 'n' is maximum $\exists$ a no. 'p' such that prob. that the process comes out of transient class ($T \geq P$)

$\therefore$ The prob that the process will not come out of the set 'T' in 'n' steps is less than (or) equal to $(1-P)$ (ie) $T \le 1-P$

ie $$\sum_{j=T} P_{ij}^{(n)} \le 1-P$$

Now consider for any $k > 0$

$$\sum P_{ij}^{(n)} = \sum_{r_1} \sum_{r_2} \dots \sum_{r_k} P_{ir_1}, P_{r_1 r_2} \dots P_{r_k j}$$

$$\le (1-P)^k \to 0 \text{ as } k \to \infty$$

Hence the proof.

Statement (iii):

In a finite M.C all the states cannot be transient.

Proof:

The statement follows from the fact that for any transient state j

$$\lim_{n \to \infty} P_{ij}^{(n)} = 0 \text{ ——— } (1)$$

and in a transition prob. matrix

$$\sum_{j=1}^{n} P_{ij}^{(n)} = 1 \text{ ——— } (2)$$

from (1) & (2), all the states cannot be transient.

Generating function Relation:

$$P_{jj}(s) = \frac{1}{1 - F_{jj}(s)} \quad \text{—— (1)}$$

$$P_{ij}(s) = F_{ij}(s)\, P_{jj}(s) \quad \text{—— (2)}$$

Proof:

(i) let us consider,

$$P_{ij}^{(n)} = \sum_{r=0}^{n} f_{ij}^{(r)} P_{jj}^{(n-r)}$$

Taking $\sum_{n=1}^{\infty} s^{(n)}$ on both sides,

$$\sum_{n=1}^{\infty} P_{ij}^{(n)} s^{(n)} = \sum_{n=1}^{\infty} \sum_{r=0}^{n} f_{ij}^{(r)} P_{jj}^{(n-r)} s^{(n)}$$

$$= \sum_{r=0}^{n} f_{ij}^{(r)} s^{(r)} \sum_{n=1}^{\infty} P_{jj}^{(n-r)} s^{(n-r)}$$

$\{\because$ Add & subtract $s^{(r)}\}$

$$= F_{ij}(s)\, P_{jj}(s) \quad \text{—— (3)}$$

consider,

$$\sum_{n=0}^{\infty} P_{ij}^{(n)} s^{(n)} = P_{ij}^{(0)} + \sum_{n=1}^{\infty} P_{ij}^{(n)} s^{(n)}$$

when $i = j$,

$$\sum_{n=0}^{\infty} P_{jj}^{(n)} s^{(n)} = P_{jj}^{(0)} + \sum_{n=1}^{\infty} P_{jj}^{(n)} s^{(n)}$$

$$P_{jj}(s) = 1 + \sum_{n=1}^{\infty} P_{jj}^{(n)} s^{(n)} \quad \left\{\because P_{jj}^{(0)} = 1\right\}$$

$$P_{jj}(s) - 1 = \sum_{n=1}^{\infty} P_{jj}^{(n)} s^{(n)} \quad \text{——} \quad (4)$$

using (3),

$$P_{jj}(s) - 1 = F_{jj}(s) P_{jj}(s)$$

$$P_{jj}(s) - F_{jj}(s) P_{jj}(s) = 1$$

$$P_{jj}(s) \left[1 - F_{jj}(s) \right] = 1$$

$$P_{jj}(s) = \frac{1}{1 - F_{jj}(s)}$$

(ii) Let us consider,

$$P_{ij}^{(n)} = \sum_{r=0}^{n} f_{ij}^{(r)} P_{jj}^{(n-r)}$$

Taking $\sum_{n=0}^{\infty} s^{(n)}$ on both sides

$$\sum_{n=0}^{\infty} P_{ij}^{(n)} s^{(n)} = P_{ij}^{(0)} + \sum_{n=1}^{\infty} P_{ij}^{(n)} s^{(n)}$$

when $i \neq j$

$$\sum_{n=0}^{\infty} P_{ij}^{(n)} s^{(n)} = 0 + \sum_{n=1}^{\infty} P_{ij}^{(n)} s^{n} \quad \left\{ \because i \neq j, \ P_{ij}^{(0)} = 0 \right\}$$

$$\Rightarrow P_{ij}(s) = F_{ij}(s) P_{jj}(s)$$

Theorem:

The state 'i' is recurrent (or) transient according as $\sum_{n=0}^{\infty} P_{ii}^{(n)} = \infty$ (or) $< \infty$

Proof:

If 'i' is recurrent

$$\sum_{n=1}^{\infty} f_{ii}^{(n)} = f_{ii}^{*} = 1$$

we have

$$P_{ii}(s) = \frac{1}{1 - F_{ii}(s)} \quad \text{—— (1)}$$

$$F_{ii}(s) = \sum_{n=1}^{\infty} f_{ii}^{(n)} s^{(n)}$$

$$\lim_{s \to 1-} F_{ii}(s) = \lim_{s \to 1-} \sum_{n=1}^{\infty} f_{ii}^{(n)} s^{(n)}$$

$$= \sum f_{ii}^{(n)} = f_{ii}^{*} = 1$$

from (1)

$$\lim_{s \to 1-} P_{ii}(s) = \frac{1}{1 - \lim_{s \to 1-} F_{ii}(s)}$$

$$= \frac{1}{1-1} = \frac{1}{0} = \infty \quad \left\{ \because \lim_{s \to 1} F_{jj}(s) = 1 \right.$$

$\Rightarrow$ 'i' is recurrent.

Note:

If 'i' is transient, then

$$\sum_{n=1}^{\infty} f_{ii}^{(n)} = f_{ii}^{*} < 1$$

$$\underset{s\to 1-}{\text{lt}}\ P_{ii}(s) = \frac{1}{1 - \underset{s\to 1-}{\text{lt}}\ F_{ii}(s)} = +ve$$

$$\underset{s\to 1-}{\text{lt}}\ \sum P_{ii}^{(n)} s^{(n)} < \infty, \quad \sum P_{ii}^{(n)} > 0$$

$$\text{(ie) } \sum P_{ii}^{(n)} < \infty$$

$\therefore$ 'i' is transient.

Theorem:

If state 'j' is persistent, non-null then as $n \to \infty$

$$\text{(i) } P_{jj}^{(nt)} \to \frac{t}{\mu_{jj}}$$

when state j is periodic with period t

$$\text{(ii) } P_{jj}^{(n)} \to \frac{1}{\mu_{jj}}$$

when state j is aperiodic.

In case state 'j' is persistent, null (whether periodic or aperiodic) then

$$P_{jj}^{(n)} \to 0 \text{ as } n \to \infty$$

Proof:

Let state j be persistent then

$$\mu_{jj} = \sum_n n \cdot f_{jj}^{(n)}$$

since $P_{jj}^{(n)} = \sum_{r=0}^{n} f_{jj}^{(r)} P_{jj}^{(n-r)}$ holds,

we may put $f_{jj}^{(n)}$ for f_n, $P_{jj}^{(n)}$ for

μ_n and μ_{jj} for μ in the lemma, we h

$$P_{jj}^{(nt)} \rightarrow \frac{t}{\mu_{jj}} \text{ as } n \to \infty.$$

when state j is periodic with period t.

when state j is aperiodic ie $(t=1)$ then

$$P_{jj}^{(n)} \longrightarrow \frac{1}{\mu_{jj}} \text{ as } n \to \infty$$

In case state 'j' is persistent null $\mu_{jj} = \infty$ and $P_{jj}^{(n)} \longrightarrow 0$ as $n \to \infty$.